The continuously critical turbulent boundary layer

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The problem of the turbulent boundary layer in two-dimensional, incompressible flow, which is continuously critical, i.e. on the point of separating, over part of its length is solved by combination of the Buri separation criterion with the boundary-layer momentum equation. In the latter an attempt is made to allow for transverse pressure variation and Reynolds stresses by over-estimating the contribution of skin friction. The Buri constant is taken to have a value of $\Gamma = -0{\cdot}04$. Whereas Stratford and Townsend assumed that an initially non-critical boundary layer could be transformed instantaneously into a critical state by the application of an infinite pressure gradient at a point, in the present approach it is considered necessary to induce a critical state over a finite length, this state then being maintained downstream. These solutions are thus not comparable near the starting-point of the flow, but fair agreement is suggested at large distances downstream, with Stratford and Townsend's theoretical solutions and with Stratford's experimental data.

Comparison is also made with an ideal pressure distribution suggested by Stuart and the basis of optimization is discussed. It is suggested that the optimum flow might be similar in form to the continuously critical flow. The thesis that a continuously critical boundary-layer flow is ideal for large extents of the suction surface of an aerofoil is denied.

The paper concludes that the Buri criterion is valid but that in this case the Buri constant has a value of $-0{\cdot}04$ rather than the accepted value of $-0{\cdot}06$. Previous work by the author in connexion with cascade aerofoils is justified. The analysis predicts theoretical minimum diffusion lengths for given pressure rises provided that the initial boundary layer has achieved a critical condition.

1. Introduction

Current interest (Allan 1961) in the theoretical limitation of the performance of aerofoils in cascade has led to consideration of the case of a turbulent boundary layer which is continuously critical, that is, at the point of separation without in fact separating, over the whole or part of its length. This case is taken to infer the shortest possible distance to achieve a desired diffusion since it also implies the maximum rate of diffusion without separation.

The problem involves the prediction of a two-dimensional, incompressible flow, pressure distribution which, when applied downstream of a point $x = x_0$, induces a continuously critical turbulent boundary layer, the layer having developed upstream of $x = x_0$ in such a way that it has a fully developed, turbulent form, and has achieved a critical condition at $x = x_0$. Stratford (1959*a*) and Townsend (1960) presented solutions for the case of a non-critical boundary layer approaching a point $x = x_0$ where it instantaneously became critical, and beyond this point became continuously critical. These solutions are therefore comparable with the results of the present analysis at points well downstream of $x = x_0$ but not near the point $x = x_0$.

By detailed analysis of the internal mechanism of the boundary-layer flow in an adverse pressure gradient, Stratford produced a two-part solution which was continuous in pressure and in pressure gradient, but discontinuous in momentum thickness. The main feature of this solution was a rapidly reducing pressure gradient from infinity at $x = x_0$, with a corresponding rapid growth of the boundary layer. Townsend introduced refinements to the first part of Stratford's solution and predicted an initial pressure distribution that did not differ seriously from Stratford's.

In a second interesting paper, Stratford (1959*b*) described an experiment in which the turbulent boundary layer on one wall of a rectangular-sectioned duct was brought very near to the continuously critical state by a painstaking development of the flow area distribution. The pressure distribution obtained was found to be very close to that predicted by his analysis.

It is also of interest in this context to consider the results of an analysis presented by Stuart (1955) based on experimental results obtained by Hewson (1949), which indicated that the transverse pressure gradient across the turbulent boundary layer was no longer negligible when the gradient of moment thickness with distance exceeded a value of one in a hundred ($d\theta/dx > 0{\cdot}01$). Stuart deduced the form of the pressure distribution required to induce a constant rate of boundary-layer growth, equal to Hewson's limiting value, and suggested that this was the 'ideal' form, permitting the maximum pressure rise over an aerofoil surface without the danger of separation.

In this paper an alternative treatment of the problem of the continuously critical turbulent boundary layer is presented which, though less fundamental in approach than that of Stratford and Townsend, produces a solution that is interesting in comparison with other results, both theoretical and experimental.

2. Analysis

The complete momentum equation for the flow in a turbulent boundary layer is

$$\frac{d\theta}{dx}+(H+2)\frac{\theta}{U}\frac{dU}{dx}=\frac{\tau_w}{\rho U^2}-\frac{1}{\rho U^2}\frac{d}{dx}\int_0^\delta (P-p)\,dy+\frac{1}{U^2}\frac{d}{dx}\int_0^\delta \overline{u'^2}\,dy, \qquad (1)$$

where H is the form factor, τ_w the wall stress, P is the static pressure at $y = \delta$ and p is the pressure in the boundary layer at a distance y from the wall, u' is the turbulent component of the velocity at y. The final two terms account for transverse pressure variation and for the Reynolds stresses, which may only be ignored if the boundary layer is far from separation and the wall curvature is small.

A continuously critical boundary layer implies zero skin friction, $\tau_w = 0$, but requires inclusion of the remaining terms on the right-hand side of equation (1).

According to Spence (1956), an accurate prediction of the separation point may be made if a flat plate relation is assumed for skin friction and the remaining terms omitted, thus compensating for their loss by over-estimating τ_w. It is convenient to adopt this procedure, taking Prandtl's (1921) relationship

$$\frac{\tau_w}{\rho U^2} = \frac{\alpha}{(\theta U/\nu)^{\frac{1}{4}}}, \tag{2}$$

where $\alpha = 0{\cdot}012 = \text{const.}$

The simplified momentum equation then becomes

$$\frac{d\theta}{dx} + (H+2)\frac{\theta}{U}\frac{dU}{dx} = \frac{\alpha}{(\theta U/\nu)^{\frac{1}{4}}}, \tag{3}$$

which was used to predict the onset of separation in cascade blades by Schlichting (1959). The critical value of the form factor H is generally taken in the range 1·8 to 2·6, so that for the continuously critical boundary layer a constant value of $H = 2{\cdot}0$ might be assumed, in accordance with Stratford's assumption in the second part of his solution.

In order to solve equation (3) with a non-zero value of α, it is necessary to introduce a separation criterion such as that of Buri (1931),

$$\frac{\theta}{U}\frac{dU}{dx} = \frac{\Gamma}{(\theta U/\nu)^{\frac{1}{4}}}, \tag{4}$$

where Γ is a constant for a critical turbulent boundary layer. Carter (1959) indicated that the Buri parameter was the basis of a variety of limiting diffusion factors in current use in cascade design and its validity has therefore been demonstrated by wide application. Experimental data on diverging channels obtained by Nikuradse (1929) indicated the critical value of Γ to be of the order of $-0{\cdot}06$.

From equations (3) and (4), it may be shown that

$$\theta/\theta_0 = (U/U_0)^n, \tag{5}$$

where $n = (\alpha/\Gamma) - (H+2) = \text{a constant}$.

Eliminating the momentum thickness from equation (4), integrating, and applying the condition that $U = U_0$ at $x = x_0$ leads to the solution

$$x - x_0 = \frac{4\theta_0 R_{\theta_0}^{\frac{1}{4}}}{(5n+1)\Gamma}\left[\left(\frac{U}{U_0}\right)^{(5n+1)/4} - 1\right], \tag{6}$$

where $R_{\theta_0} = U_0\theta_0/\nu$, and x_0 is the equivalent length of constant-pressure turbulent boundary layer required to develop a momentum thickness θ_0.

The relation between θ_0 and x_0 may be taken from Goldstein (1938) or Schlichting (1955) to be

$$\theta_0 = 0{\cdot}036 x_0 R_0^{-\frac{1}{5}}, \tag{7}$$

where $R_0 = U_0 x_0/\nu$, which, when substituted into equation (6) produces

$$\frac{x}{x_0} - 1 = \frac{0{\cdot}0628}{(5n+1)\Gamma}\left[\left(\frac{U}{U_0}\right)^{(5n+1)/4} - 1\right]. \tag{8}$$

Taking $\alpha = 0{\cdot}012$ and $H = 2{\cdot}0$, the choice of Buri constant remains. The Nikuradse value $\Gamma = -0{\cdot}06$ leads to

$$(x/x_0) - 1 = 0{\cdot}0523[(U/U_0)^{-5} - 1]. \tag{9}$$

This solution was found to compare unfavourably with Stratford's results and consequently a value of $\Gamma = -0{\cdot}04$ was considered, when

$$(x/x_0) - 1 = 0{\cdot}0766[(U/U_0)^{-5{\cdot}125} - 1]. \tag{10}$$

These solutions are compared with Stratford's and with Stuart's theoretical results, and with Stratford's experimental data in the following sections.

3. Initial flow condition

At this point it is pertinent to comment upon the amendment in this paper of the original problem examined by Stratford and Townsend. The latter assumed that the boundary layer would be capable of changing instantaneously from a non-critical state to a critical condition and this led them to prescribe an infinite pressure gradient at $x = x_0$, inducing an infinite rate of boundary-layer growth at that point. In the present analysis the upstream flow is assumed to vary in such a way that the boundary layer has become critical, but is unseparated, at $x = x_0$, this condition being achieved over an unspecified length. The instantaneous change of boundary-layer condition is evidently an extreme case in the present analysis, when the length to achieve the critical condition is reduced to zero. The theoretical pressure gradient required is then infinite at the point $x = x_0$. In practice such a gradient is only achieved in flow around a sharp corner or through a shock, and in general would involve the danger of separation of the boundary layer, with the exception of a limited range of flow systems in which it might be possible to induce a weak shock, when separation might be avoided. It is suggested therefore that an instantaneous change of boundary-layer condition is, in general, an impracticable requirement and that the critical condition must be achieved over an arbitrary, finite length.

Further it is suggested that if the pressure distribution predicted by the present analysis were applied to a turbulent boundary layer that was not initially critical, then the layer would not immediately approach separation but might become critical at some point downstream, becoming continuously critical beyond that point. In this context, the present analysis predicts a safe, practical solution.

An exception to the previous argument is the case of a continuously critical turbulent boundary layer over the whole of the suction surface of an aerofoil, considered by the author in 1961. Since, near the leading-edge stagnation point, the boundary layer is of negligible thickness, a very large pressure gradient may be applied near that point without risk of separation, provided that the gradient is rapidly alleviated downstream of the stagnation point. In his previous work the author assumed $\Gamma = -0{\cdot}06$ and deduced a velocity distribution

$$U/U_2 = (x/c)^{-\frac{1}{5}}, \tag{11}$$

where x is the surface distance aft of the leading edge stagnation point, c the chord length of the aerofoil, U the local free stream velocity and U_2 the exit velocity from the cascade. Had a value of $\Gamma = -0{\cdot}04$ been employed, then the velocity distribution would have been

$$U/U_2 = (x/c)^{-1/5{\cdot}125}, \tag{12}$$

and it appears that the solution was not very sensitive to variation of the Buri constant.

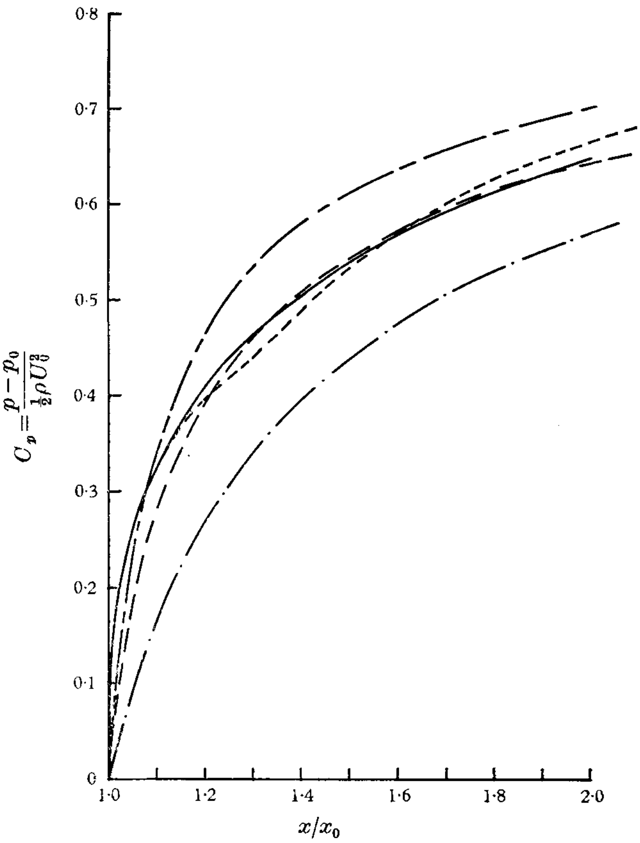

FIGURE 1. Comparison of pressure distributions: ——, Stratford's theory; ----, Stratford's experimental data; — — —, present theory with $\Gamma = -0{\cdot}04$; ——, present theory with $\Gamma = -0{\cdot}06$; —·—, Stuart's theory.

4. Comparison with Stratford's theoretical solution

In view of the disagreement in initial boundary-layer condition at $x = x_0$, the present solution cannot be expected to compare favourably with that of Stratford immediately downstream of $x = x_0$. It might be anticipated, however, that the solutions should become asymptotic at a large distance downstream from the starting-point, where the influence of the starting condition would be reduced to a lesser proportion of the pre-history of the layer.

The solutions are compared for pressure distribution, pressure gradient variation and boundary-layer growth in figures 1, 2 and 3 respectively. Com-

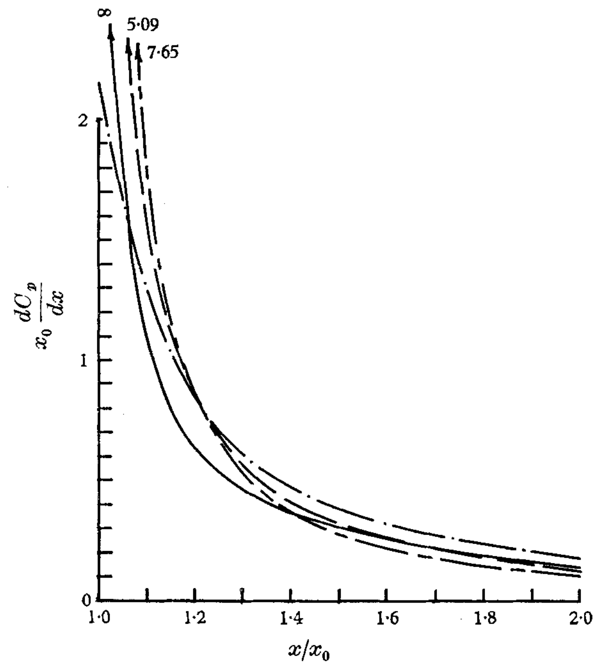

FIGURE 2. Distribution of pressure gradients: ——, Stratford's theory; —·—, Stuart's theory; ———, present theory with $\Gamma = -0{\cdot}04$; ———, present theory with $\Gamma = -0{\cdot}06$.

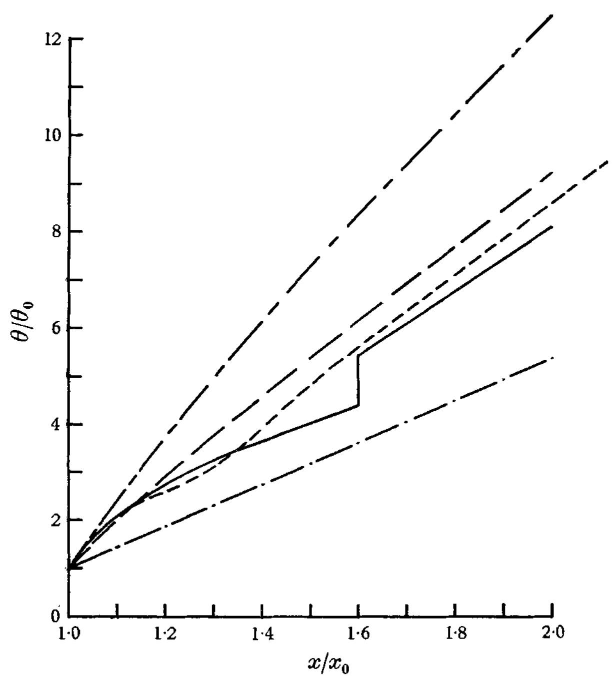

FIGURE 3. Development of momentum thickness: ——, Stratford's theory; ----, Stratford's experimental data; —·—, Stuart's theory; ———, present theory with $\Gamma = -0{\cdot}04$; ———, present theory with $\Gamma = -0{\cdot}06$.

parison of the pressure distributions suggests that the normally accepted value of $\Gamma = -0{\cdot}06$ is invalid for the continuously critical flow, though not necessarily for a flow in which the boundary layer approaches a critical value at a point. In the latter case the value of the Buri criterion changes very rapidly near separation so that variations of the Buri constant in the range $-0{\cdot}04$ to $-0{\cdot}06$ would make little difference to the estimated location of the separation point. The predicted pressure distribution for $\Gamma = -0{\cdot}04$ is in good agreement with that of Stratford for values of $x/x_0 > 1{\cdot}3$. A very rapid alleviation of pressure gradient is associated with Stratford's infinite initial value while the present analysis suggests a more gradual alleviation of the finite initial pressure gradient. For values of $x/x_0 > 1{\cdot}4$, pressure gradients for both $\Gamma = -0{\cdot}06$ and $\Gamma = -0{\cdot}04$ are in good agreement with Stratford's values.

The higher initial pressure gradients of the present analysis are reflected in the growth of momentum thickness in figure 3, which also indicates the discontinuity in Stratford's solution at $x/x_0 = 1{\cdot}6$ where $C_p = 4/7$. This diagram demonstrates the excessive boundary-layer growth suggested by the present analysis for $\Gamma = -0{\cdot}06$ and indicates the improvement in the comparison for $\Gamma = -0{\cdot}04$.

Townsend's analysis suggested a pressure distribution that was slightly steeper than Stratford's immediately downstream of $x = x_0$ but which was substantially in agreement with Stratford's result.

Both Stratford's theory and the present solution suggest that the pressure distributions of figure 1 are insensitive to initial Reynolds number R_0, the latter disappearing in the present approach, and in Stratford's solution $C_p \propto (R_0)^{1/15}$.

5. Comparison with Stratford's experimental data

Stratford's experiment produced a flow that was almost continuously critical for $1 \leqslant x/x_0 \leqslant 2{\cdot}0$, though not exactly critical even by Stratford's assessment, which compared two integrated terms of the momentum equation which are not necessarily equal for zero friction flow when Reynolds stresses and transverse pressure gradients, which were in fact detected, are significant. Nevertheless, the experiment was highly successful in view of the difficulty incurred by the use of a low aspect ratio flow surface and the consequent danger of corner separation which required the application of corner suction. The results of this experiment provide an invaluable guide in the evaluation of theoretical results.

The experimental pressure distribution was in fair agreement with that predicted by Stratford and Townsend. Unfortunately, a short relief of pressure gradient occurred near $x/x_0 = 1{\cdot}3$ and beyond this point the flow became increasingly further from separation. Thus a direct comparison of pressure distributions in the region of greatest interest, $1{\cdot}3 \leqslant x/x_0 \leqslant 2{\cdot}0$, is not possible but it might be suggested that the present prediction for $\Gamma = -0{\cdot}04$ appears reasonable in this region, bearing in mind that the two curves are not comparable near $x = x_0$.

The short relief in the experimental pressure distribution is reflected in the growth of momentum thickness in figure 3, where it might be anticipated that, had the boundary layer been truly critical, the momentum thickness would have been greater than that measured, and, therefore, would have been nearer to the predicted line for $\Gamma = -0{\cdot}04$. Some evidence for this proposition was provided by Stratford, who found that by a small increase in divergence of the channel an increase in displacement thickness, and also presumably momentum thickness, of 10 % could be achieved without causing backflow, with no appreciable alteration of the pressure distribution, which confirms the statement that the flow as not exactly critical. In figure 3 it is evident that an increase of 10 % in the experimental momentum thickness would produce an excellent correlation between this data and the predicted growth for $\Gamma = -0{\cdot}04$.

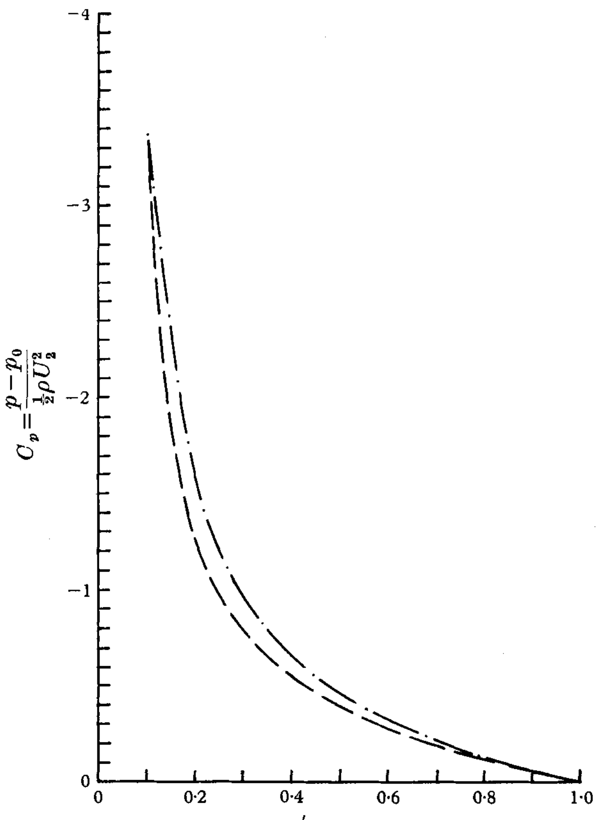

FIGURE 4. Comparison with Stuart's ideal pressure distribution on the surface of an aerofoil: —·—, Stuart's theory; ———, present theory with $\Gamma = -0{\cdot}04$.

6. Comparison with Stuart's theoretical solution

Comparative data from Stuart's analysis is presented in figures 1, 2 and 3 for the case of diffusion in a duct, and in figure 4 for diffusion on the surface of an aerofoil downstream of a point at a distance of 10 % of chord behind the leading edge. In the former case Stuart's diffusion rates appear to be considerably less than those achieved in Stratford's experiment. In the case of the aerofoil, the present approach suggests a velocity distribution defined by

$$\bar{U}^{-5{\cdot}125} - 1 = k(\bar{x} - 1), \tag{13}$$

where $\bar{U}$ is the ratio of the local velocity to the trailing edge velocity, $\bar{x}$ the ratio of the distance aft of the leading edge to the chord length, and k a constant determined by the velocity at and the location of the beginning of diffusion. The Buri constant has been taken to be $\Gamma = -0{\cdot}04$. For the case considered in figure 4, $k = 1{\cdot}088$. Again it is found that Stuart's diffusion rates were comparatively low.

This comparison might have been anticipated in view of the definition of Hewson's limiting value of $d\theta/dx = 0{\cdot}01$. The suggestion that transverse pressure gradients would only begin to become appreciable if this limit were exceeded implies that larger rates of boundary-layer growth, and therefore higher diffusion rates, would be achieved before the onset of separation. Stuart further suggested that the simplified momentum equation (3) became increasingly inaccurate beyond Hewson's limiting value, that is, near separation. This appears to contradict the experience of Spence, mentioned earlier.

Stratford also examined the theory of von Doenhoff & Tetervin (1943) when applied to the problem of a continuously critical turbulent boundary layer, and found that the theory was in marked disagreement with his experimental results.

7. The ideal pressure distribution

It is evident that an 'optimum' flow should satisfy the requirement that the rate of increase of momentum thickness (or energy thickness) with change of pressure should be a minimum. Considering the range of diffusion flows bounded by the extreme cases of constant pressure flow on the one hand and maximum diffusion flow on the other, for the former ($dC_p = 0$) the gradient $d\theta/dC_p$ has an infinite value, whereas for the latter $d\theta/dC_p$ will be finite. It might be anticipated therefore that the optimum flow will be similar to the continuously critical flow, involving high initial pressure gradients. This conclusion is confirmed to some extent by experimental results presented by Schubauer & Spangenberg (1960). Stratford observed that the momentum equation was inconclusive in this argument due to the opposing behaviour of the skin friction and pressure gradient terms, but suggested that the energy equation indicated a minimum loss in the case of continuously critical flow. The theoretical evaluation of the problem depends upon the complex relationship between skin friction and pressure gradient and may also involve the influence of transverse gradients and Reynolds stresses.

Stuart defined the 'ideal' pressure distribution over the surface of an aerofoil as that which would induce the maximum diffusion over that surface, that is, a continuously critical boundary-layer flow. Such a flow might usefully be applied near the trailing edge of the suction surface of an aerofoil, as envisaged by Stratford, though in practice a safety margin might be desirable to avoid the danger of separation occurring instantaneously over a large portion of the surface in the stalled condition. However, the primary requirement of the aerofoil is that it should develop a lift force with the greatest possible efficiency, which may be represented by lift to drag ratio. Although the over-all diffusion obtained over the surface of an aerofoil on which the flow is continuously critical for a large part of the length is very high, the very rapid reduction of suction pressure induces a low lift coefficient in conjunction with a maximum rate of boundary-layer growth with distance, and therefore a low lift-to-drag ratio. This argument was demonstrated by Allan (1961), when it was shown that optimum velocity distributions induced critical boundary-layer conditions at or near the trailing edge in association with a constant velocity over the forward portion of the surface.

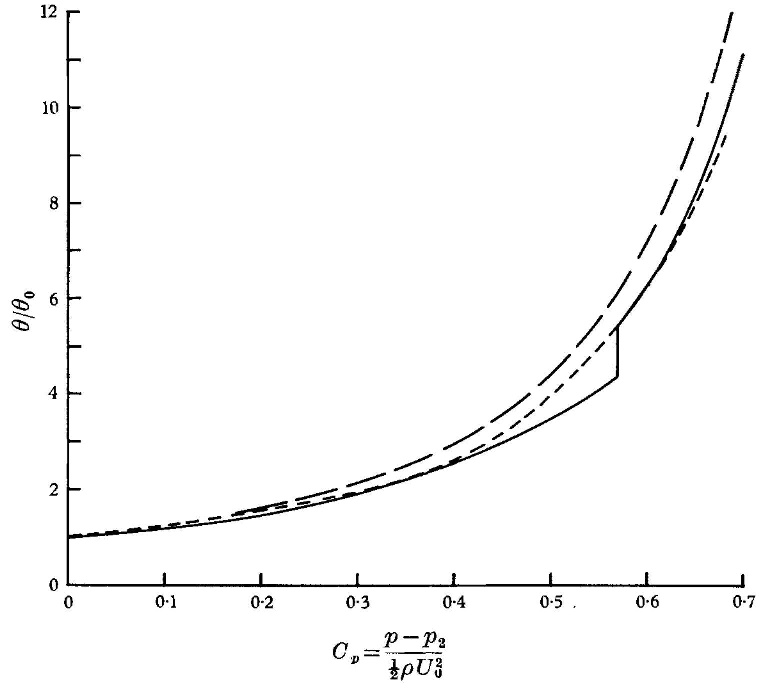

FIGURE 5. Variation of momentum thickness with pressure coefficient: ——, Stratford's theory; ----, Stratford's experimental data; — — —, present theory with $\Gamma = -0{\cdot}04$.

8. Conclusions

The problem of the generation of a continuously critical turbulent boundary layer downstream of a point at which critical conditions have been achieved has been analysed by a method which, though less fundamental than those of Stratford and Townsend, produces a solution that has the advantage of a continuous development of momentum thickness. Though direct comparison is not possible, agreement with Stratford's theoretical solution and with his measured data is suggested in a region well downstream of the starting-point $x = x_0$.

The validity of the Buri criterion for separation has been demonstrated, though for the flow considered it was found that the Buri constant, Γ, should be taken to be $-0{\cdot}04$ rather than Nikuradse's value of $-0{\cdot}06$. It was also suggested that this modification had little effect on the velocity distribution for a continuously critical turbulent boundary layer on the suction surface of an aerofoil, previously discussed by the author.

In a discussion of Stuart's analysis, which did not imply a critical boundary layer, and could not therefore predict maximum diffusion rates, it was suggested that the optimum pressure distribution would induce the minimum gradient of momentum or energy thickness with flow velocity or pressure, and that the form of this distribution would be similar to that for continuously critical flow.

The analysis presented in this paper provides a simple means of estimation of minimum diffusion lengths for desired pressure rises provided that the boundary layer is brought to a critical condition in a short, but finite distance. In practice the achievement of these lengths must be complicated by the problems of three-dimensional flow, but Stratford has show that the two-dimensional flow is unexpectedly stable.

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